Euler's Science of Combinations

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In addition to all the other mathematics he did, Euler discovered the first significant results in many fields of modern combinatorics. In this chapter, we survey this work spread over some fourteen publications.

In the first section, we consider Euler's work on partitions of integers, focusing on three articles and a book that span his career. The second section addresses various types of squares that Euler considered — magic, Graeco-Latin, and chessboards. The final section samples Euler's contributions to the study of binomial coefficients, the Catalan numbers, derangements, and the Josephus problem. We omit the bridges of Königsberg and the polyhedron formula as they are treated elsewhere in this volume..

1. Partitions

In 1674 Leibniz wrote to J. Bernoulli asking about "divulsions of integers," now called partitions. A basic problem is determining the number $p(n)$ of ways that a positive integer n can be written as the sum of positive integers; for example, $p(4) = 5$, corresponding to the sums 4, 3 + 1, 2 + 2, 2 + 1 + 1 and 1 + 1 + 1 + 1. Variations of this basic problem ask for partitions of n into a given number of parts, or into distinct parts, odd parts, etc. For example, we can write the number 10 as the sum of exactly three positive numbers in eight ways,

$$\begin{array}{cccc} 8+1+1 & 7+2+1 & 6+3+1 & 6+2+2 \\ 5+4+1 & 5+3+2 & 4+4+2 & 4+3+3 \end{array}$$

Notice that four of these are partitions with distinct parts.

The first publication on partitions of integers came from a presentation that Euler made in 1741 to the St. Petersburg Academy [E158]. Euler answered two questions posed by Philip Naudé and stated what became known as the pentagonal number theorem. We present his arguments from a later publication, his celebrated *Introductio in Analysin Infinitorum* [E101], in which his main method of proof is generating functions; Euler often repeated his results on partitions, in some cases providing multiple proofs.

Naudé's Question 1: In how many ways can the number 50 be written as the sum of seven distinct positive integers? To answer this, Euler considered the following infinite product in x and z, organized in increasing powers of z.

$$\begin{aligned}(1+xz)\ &(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)(1+x^6z)\cdots \\ &= 1+z(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\cdots) \\ &\quad +z^2(x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+4x^{10}+\cdots) \\ &\quad +z^3(x^6+x^7+2x^8+3x^9+4x^{10}+5x^{11}+7x^{12}+8x^{13}+\cdots)+\cdots\end{aligned} \tag{1}$$

What does a term such as $4x^{10}z^3$ indicate? Each $x^{10}z^3$-term arises from one of the four products $x^7z \cdot x^2z \cdot xz$, $x^6z \cdot x^3z \cdot xz$, $x^5z \cdot x^4z \cdot xz$, and $x^5z \cdot x^3z \cdot x^2z$. These products correspond to the above four ways that we can write 10 as a partition of three distinct positive integers.

However, we do not want to have to compute the coefficient of $x^{50}z^7$ from the terms of this infinite product. Writing $m^{(\mu)i}$ for the number of ways of writing m as the sum of μ "inequal" integers, Euler established the following recurrence relation:

$$(m+\mu)^{(\mu)i} = m^{(\mu)i} + m^{(\mu-1)i}$$

With this, it is not hard to compute 522 as the answer to Naudé's first question.

Naudé's Question 2: In how many ways can the number 50 be written as the sum of seven positive integers, equal or unequal? Here Euler considered another infinite product in x and z, this time with factors in the denominator.

$$
\begin{aligned}
&\frac{1}{(1-xz)(1-x^2z)(1-x^3z)\cdots}\\
&\quad=\left(\frac{1}{1-xz}\right)\left(\frac{1}{1-x^2z}\right)\left(\frac{1}{1-x^3z}\right)\cdots\\
&\quad=(1+xz+x^2z^2+x^3z^3+\cdots)(1+x^2z+x^4z^2+x^6x^3+\cdots)\cdots \qquad (2)\\
&\quad=1+z(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+\cdots)\\
&\qquad+z^2(x^2+x^3+2x^4+2x^5+3x^6+3x^7+4x^8+4x^9+\cdots)\\
&\qquad+z^3(x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+\cdots)+\cdots
\end{aligned}
$$

Here, a term such as $8x^{10}z^3$ indicates the eight ways (given above) that we can write 10 as the sum of three positive integers. Again, there is a recurrence relation. Writing $m^{(\mu)}$ when the parts need not be distinct, Euler established the following equation:

$$m^{(\mu)}=(m-\mu)^{(\mu)}+(m-1)^{(\mu-1)}$$

From this, we can determine that the answer to Naudé's second question is 8496. But Euler first established a connection between the two questions. He deduced the number of partitions with μ distinct parts from the formula

$$m^{(\mu)i}=\left(m-\frac{\mu(\mu-1)}{2}\right)^{(\mu)}$$

— in particular, the number of unrestricted seven-part partitions of 50 is equal to the number of distinct seven-part partitions of $50+21=71$. Euler also discussed the connection between the numbers of parts in a partition and the maximum number of parts; for example, 8496 is also the number of partitions of $50-7=43$ that use only the numbers $1,2,\ldots,7$.

This revolutionary paper ends with a celebrated formula that Euler had mentioned in his correspondence (see [B]), but had not yet proved. If we let $z=1$ in equation (2) we can combine the expressions in x to give

$$
\begin{aligned}
&\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}\\
&\quad=1+x+2x^2+3x^3+5x^4+7x^6+11x^7+15x^8+22x^9+\cdots
\end{aligned}
$$

where the coefficient of x^k is the total number of unrestricted partitions of k. But now consider the reciprocal of this infinite product. From extensive computations Euler concluded that

$$
\begin{aligned}
P&=(1-x)(1-x^2)(1-x^3) \qquad (3)\\
&=1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+\cdots
\end{aligned}
$$

where the exponents are the generalized pentagonal numbers, $(3k^2 \pm k)/2$. This result is now known as Euler's pentagonal number theorem.

Euler devoted a chapter of his 1748 *Introductio* [E101] to partitions, expanding on the results from the previous article. It includes one of the most striking and elegant applications of generating functions to partitions. Letting $z = 1$ in equation (1) and combining the expressions in x we obtain

$$\begin{aligned} Q &= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots \\ &= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \cdots \end{aligned}$$

where the coefficient of x^k is the total number of partitions of k into distinct parts. Again, we consider the reciprocal of this infinite product. With the infinite product P as defined in equation (3), we note that the terms of

$$PQ = (1-x^2)(1-x^4)(1-x^6)\cdots$$

are factors of P, so that we can divide P by PQ. This leaves

$$\frac{P}{PQ} = \frac{(1-x)(1-x^2)(1-x^3)\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots} = (1-x)(1-x^3)(1-x^5)\cdots = \frac{1}{Q}$$

so that Q can now be written as

$$\begin{aligned} Q &= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} \\ &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\cdots \\ &= (1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)\cdots \\ &= (1 + x + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots)\cdots \end{aligned}$$

in which the coefficient of x^k gives the number of partitions of k into odd parts, not necessarily distinct. This proves a surprising theorem:

The number of partitions of k into distinct parts equals the number of partitions of k into odd parts.

As an example, note that there are six partitions of the number 8 into distinct parts (8, 7 + 1, 6 + 2, 5 + 3, 5 + 2 + 1, and 4 + 3 + 1) and six partitions of 8 into odd parts (7 + 1, 5 + 3, 5 + 1 + 1 + 1, 3 + 3 + 1 + 1, 3 + 1 + 1 + 1 + 1 + 1, and 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1).

The chapter concludes with generating-function proofs of the facts that each positive integer can be expressed uniquely as a sum of distinct powers of 2 and can also be uniquely expressed as a sum and difference of distinct powers of 3.

Euler continued his exploration of partitions with a paper presented in early 1750 [E191]. This is his longest article on partitions, filled not so much with new material, but rather with numerous examples and tables. Writing $n^{(\infty)}$ for what is now known as $p(n)$, he established that

$$n^{(\infty)} = (n-1)^{(1)} + (n-2)^{(2)} + (n-3)^{(3)} + \cdots + (n-n)^{(n)}$$

which he then used recursively to suggest a recurrence relation for $n^{(\infty)}$; this also follows from the still unproved pentagonal number theorem (3).

$$\begin{aligned} n^{(\infty)} = & (n-1)^{(\infty)} + (n-2)^{(\infty)} - (n-5)^{(\infty)} \\ & -(n-7)^{(\infty)} + (n-12)^{(\infty)} + \cdots \end{aligned}$$

Andrews [A] asserts that "No one has ever found a more efficient algorithm for computing $p(N)$. It computes a full table of values of $p(n)$ for $n \leq N$ in time $O(N^{3/2})$."

Later in 1750, in a letter to Christian Goldbach, Euler finally proved the pentagonal number theorem (3). He eventually published two proofs, and also considered properties of the pentagonal numbers themselves, such as that each pentagonal number is one-third of a triangular number; see Bell [B] for a detailed account of this pentagonal-number result throughout Euler's work. Interestingly, the function that sums the divisors of a number — for example, $\int 10 = 1 + 2 + 5 + 10 = 18$ in Euler's notation — shares almost the same recurrence relation; Euler also devoted several articles to this divisor function.

Euler returned to partitions once more in a presentation of 1768 [E394], in which he combined two previous restrictions on partitions – the number of parts and how large each part can be. The running example for much of the article used only $1, 2, \ldots, 6$ as parts, and Euler's computation of the coefficients in $(x + x^2 + \cdots + x^6)^n$ was simplified by the use of various recurrence relations; for example, Euler established that

$$n^{(6)} = \frac{(n-1)(n-1)^{(6)} - (48-n)(n-6)^{(6)} - (43-n)(n-7)^{(6)}}{n-6}$$

as an example of the formulas that can be derived from these methods.

The article also considers the problem of partitions with varying constraints. In particular, Euler considered three-part partitions where the first part is 6 or less, the second is 8 or less, and the third is 12 or less. The 576 resulting partitions of the numbers from 3 to 26 are specified by the coefficients of

$$(1 + x + \cdots + x^6)(1 + x + \cdots + x^8)(1 + x + \cdots + x^{12})$$

whose computations are simplified by generating-function techniques. The coefficients for x^k, $k = 3, 4, \ldots, 14$ are as follows: The coefficients of x^k for

exponent	3	4	5	6	7	8	9	10	11	12	13	14
coefficient	1	3	6	10	15	21	27	33	38	42	45	47

$k = 15, 16, \ldots, 26$ are the reverse of these, from 47 down to 1.

Although Euler was not the first mathematician to consider generating functions or partitions of integers — De Moivre [dM] had used generating functions in 1718 to analyze multiple-step recurrence relations — he was the first to treat them in a thorough and general way. A thorough early history of partitions, drawing on some of Euler's correspondence and including work of his contemporaries, is given in Dickson [D]. Generating functions have since become an essential tool in combinatorics and number theory, "a clothes line on which we hang up a sequence of numbers for display" (see Wilf [W]). Even though Hardy and Ramanujan obtained a stunning exact formula for the partition number $p(n)$, the theory of partitions has remained a very active area of research with many impressive results and many outstanding problems (see Andrews and Eriksson [AE]). Even so, we can still agree with Andrews [A] that "Almost every discovery in partitions owes something to Euler's beginnings."

2. Squares

In 1776, Euler delivered a short article *On magic squares* to the St. Petersburg Academy [E795]. Such an arrangement of integers, already long familiar, is an $n \times n$ square with the numbers $1, 2, \ldots, n^2$ arranged in such a way that the numbers in each row, each column, and each of the two diagonals have the same sum. After discussing what this sum must be, Euler introduced Latin and Greek letters to help him to analyze magic squares: each Latin letter stands for a multiple of n from 0 to $n(n-1)$ and each Greek letter has a value from 1 to n. With each individual cell assigned both a Latin and a Greek letter in such a way that no pair is repeated, he was able to determine values for the letters so that the sums give a magic square. An example of such a 3×3 square is given in Table 1. Note that the left-hand square has each letter occurring exactly once in each row and column, and is an example of what is now known as a Graeco-Latin square (because of Euler's notation).

A 4×4 non-Graeco-Latin square appears in Table 2, with the associated magic square obtained by using the specified values. Although the square is not Graeco-Latin (notice that α appears twice in the first column, a twice

$a\gamma$	$b\beta$	$c\alpha$
$b\alpha$	$c\gamma$	$a\beta$
$c\beta$	$a\alpha$	$b\gamma$

2	9	4
7	5	3
6	1	8

Table 1
Graeco-Latin square for $n = 3$ and associated magic square from $a = 0, b = 6, c = 3; \alpha = 1, \beta = 3, \gamma = 2$.

in the first row), it still yields a magic square. Euler's article closes with descriptions (depending on whether n is even or odd) of how to construct magic squares of any size.

$a\alpha$	$a\delta$	$d\beta$	$d\gamma$
$d\alpha$	$d\delta$	$a\beta$	$a\gamma$
$b\delta$	$b\alpha$	$c\gamma$	$c\beta$
$c\delta$	$c\alpha$	$b\gamma$	$b\beta$

1	4	14	15
13	16	2	3
8	5	11	10
12	9	7	6

Table 2
Non-Graeco-Latin square for $n = 4$ and associated magic square from $a = 0, b = 4, c = 8, d = 12; \alpha = 1, \beta = 2, \gamma = 3, \delta = 4$.

Euler remained interested in the problem of Graeco-Latin squares. Three years later, in 1779, he presented one of his longest published papers, *Investigations on a new type of magic square* [E530]. It begins with the celebrated "36-officers problem:"

Six regiments are each represented with six officers, one per rank — can they be placed in a 6 by 6 formation such that there is one officer of each regiment in each row and column, and one officer of each rank in each row and column?

Euler claimed that the answer is no, and embarked on a thorough study of Graeco-Latin and Latin squares.

By the second page, Euler had replaced his Graeco-Latin notation with pairs of numbers, the second written in superscript; we give an example in Table 3. He gave several general methods for building Latin squares, of which the "double march" is illustrated in Table 3 on the right — notice how the square divides into four smaller Latin squares involving 1 and 2 or 3 and 4. There are also single, triple, and quadruple marches.

1^1	4^3	2^4	3^2
2^2	3^4	1^3	4^1
3^3	2^1	4^2	1^4
4^4	1^2	3^1	2^3

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Table 3
Revised notation for Graeco-Latin squares, and example of a "double march."

He also used these methods (and others) to build Graeco-Latin squares of odd order and orders that are multiples of 4; however, none of the methods produced a 6×6 Graeco-Latin square. Euler suspected that there are none, claiming that he would have come across one in his investigation if any existed, while recognizing that an exhaustive search would be very lengthy. There is no formal conjecture of the general case, but he stated that a Graeco-Latin square of order $4k + 2$ would have to be "completely irregular" and seemed to doubt that there are such. Euler's article also includes enumeration of Latin squares of small orders under certain conditions and a discussion of collections of Latin squares any two of which can be combined into a Graeco-Latin square. For example, notice in Table 3 that the Latin square of base numbers on the left can be combined with the Latin square on the right to make a Graeco-Latin square, and the same is true for the Latin square of superscript numbers.

An exhaustive search in the next century verified Euler's conjecture for 6×6 Graeco-Latin squares, showing that Euler was right about the 36 officers problem. However, the general $4k + 2$ conjecture was shown to be false in 1960 by Bose, Shrikhande and Parker (see Klyve and Stemkoski [KS] for details); this result was so unexpected that it was reported on the front page of the *New York Times*. A related research area is that of finding "mutually orthogonal Latin squares:" are there $n - 1$ Latin squares of size $n \times n$ with the property that any two of them constitute a Graeco-Latin square? (The three 4×4 Latin squares of Table 3 are an example.) This is an area of contemporary research (see Mullen [M]).

Recently, there have been uninformed claims in the media that Euler invented the popular number puzzle Sudoku in which 9×9 Latin squares satisfy the additional requirement that no number should be repeated in the principal 3×3 subsquares. While a completed Sudoku puzzle is a Latin square, none of Euler's 9×9 examples of Latin squares has the form of a Sudoku puzzle. The closest he came were the 4×4 examples of Table 3 (each set of numbers in the left-hand square), which coincidentally have the additional structure that each 2×2 corner contains 1, 2, 3, and 4.

However, Euler did write on a topic in recreational mathematics that relates to squares. His paper *Solution of a curious question that does not seem to have been subject to any analysis* is based on a 1759 presentation to the Berlin Academy about knight's tours on chess boards of various sizes [E309]. The question is how to have a knight make its L-shaped moves around the board and visit each square exactly once (now known as a Hamiltonian cycle!). Euler demonstrated many such tours on standard 8×8 and other size boards, often producing tours with high degrees of symmetry. A knight's tour is shown by labeling consecutive positions, as in Tables 4 and 5.

37	62	43	56	35	60	41	50
44	55	36	61	42	49	34	59
63	38	53	46	57	40	51	48
54	45	64	39	52	47	58	33
1	26	15	20	7	32	13	22
16	19	8	25	14	21	6	31
27	2	17	10	29	4	23	12
18	9	28	3	24	11	30	5

Table 4
Closed knight's tour of an 8×8 board with half-turn symmetry about the center of the board.

For an $n \times n$ board, the labels are $1, 2, \ldots, n^2$ — could a knight's tour give rise to a magic square? This is not a question Euler posed; the closest such path given in the article is the 5×5 example of Table 5: the diagonals, as well as rows and columns including the center, all sum to 65, but this seems coincidental. Computers have recently been used to conclude that there are no 8×8 "Euler knight tours," but if the requirement about diagonal sums is removed, then there are 140 such tours (see Jelliss [J]).

7	12	17	22	5
18	23	6	11	16
13	8	25	4	21
24	19	2	15	10
1	14	9	20	3

Table 5
Knight's tour of a 5×5 board.

3. Other Topics

Binomial coefficients

Many of Euler's articles incorporate binomial coefficients; here we highlight three articles that consider properties of these numbers with integer arguments.

In a 1776 presentation to the St. Petersburg Academy [E575], primarily about integrals, Euler collected several facts about binomial coefficients, using notations very similar to those used today. One primary result, in modern notation, is the following equation:

$$\binom{n}{0}\binom{p}{q}+\binom{n}{1}\binom{p}{q+1}+\cdots=\binom{p+n}{q+n}$$

Later in the same year, Euler presented an article generalizing binomial coefficients to higher-degree polynomials [E584]. He first reviewed the relationship between

$$\binom{n}{p}$$

and the coefficients of $(1+z)^n$, and properties such as the sum of squares (a special case of the preceding formula) and

$$\binom{n+1}{p+1}=\binom{n}{p}+\binom{n}{p+1}$$

Euler then moved on to trinomial, quadrinomial, and higher-order coefficients. In particular, the coefficients of $(1+z+zz+z^3)^n$ (to use his notation for squares) for small values of n are given in the following partial table, where the columns correspond to the degree of z.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1													
1	1	1	1	1										
2	1	2	3	4	3	2	1							
3	1	3	6	10	12	12	10	6	3	1				
4	1	4	10	20	31	40	44	40	31	20	10	4	1	
5	1	5	15	35	65	101	135	155	155	135	101	65	35	15
6	1	6	21	56	120	216	336	456	546	580	546	*etc.*		

Table 6
Coefficient of z^k in $(1+z+z^2+z^3)^n$, the quadrinomial coefficients.

He showed that these coefficients, indexed here with 4, satisfy

$$\binom{n+1}{p+3}_4=\binom{n}{p+3}_4+\binom{n}{p+2}_4+\binom{n}{p+1}_4+\binom{n}{p}_4$$

and the general relationship

$$\binom{n}{0}_4 \binom{m}{0}_4 + \binom{n}{1}_4 \binom{m}{1}_4 + \cdots = \binom{n+m}{3n}_4$$

In 1778, Euler returned to these coefficients in another presentation in the same venue [E709]. By writing $(1+z+zz+z^3)^n = (1+z(1+z+zz))^n$, he related the quadrinomial coefficients to the binomial and trinomial ones. For example, again writing subscripts for the degree so that binomial coefficients are indexed by 2, we have

$$\begin{aligned}\binom{n}{4}_4 &= \binom{n}{4}_2 \binom{4}{0}_3 + \binom{n}{3}_2 \binom{3}{1}_3 + \binom{n}{2}_2 \binom{2}{2}_3 \\ &\quad + \binom{n}{1}_2 \binom{1}{3}_3 \\ &= \binom{n}{4}_2 + 3\binom{n}{3} + 3\binom{n}{2}\end{aligned}$$

Catalan numbers

In a letter of 4 September 1751 to Christian Goldbach [EG], Euler discussed the problem of finding the number of different ways that a polygon can be broken into triangles using diagonals. After considering several examples, he gave the formula

$$\frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)}$$

which we now call the $(n-2)$nd Catalan number, usually written as $\frac{1}{n-1}\binom{2n-4}{n-2}$. Euler closed his letter with the generating function associated with this sequence, a topic that he and Goldbach discussed in subsequent correspondence:

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \cdots = \frac{1 - 2a - \sqrt{1-4a}}{2a^2}$$

Derangements

Many of Euler's articles discuss probability and games of chance, especially lotteries. One that is relevant here is his *Calculation of the probability in the game of coincidence* [E201], published in 1753. Two players have identical decks of cards, shuffled, which they turn over one at a time. If they turn over the same card at any turn, the first player wins and the game ends. The second player wins only if the cards are different at every turn. Euler explained that this is equivalent to numbering the cards and checking to see if the second player turns over card n on turn n, and showed that the probability of the second player winning is

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \cdots = \frac{1}{e}$$

In 1779, Euler returned to this topic with *The solution of a curious question in the science of combinations*, presented to the St. Petersburg Academy [E738]. No longer motivated by the card game, he asked for the number of ways that the sequence a, b, c, d, e, ..., can be reordered such that no letter is in its original position. We will write $D(n)$ for this, suggesting the later name "derangement" for such an ordering. Euler derived the following two identities, and showed their equivalence:

$$D(n) = (n-1)(D(n-1) + D(n-2)) \quad \text{and}$$
$$D(n) = nD(n-1) + (-1)^n$$

The Josephus problem

We close with another topic in recreational mathematics, a staple of discrete mathematics textbooks. Suppose that n people stand in a circle. Moving clockwise, we remove every kth person. Which person is the last to be removed? This is known as the Josephus Flavius problem, named for the Jewish historian and general and an intricate suicide pact which left him the last man standing. In *Observations about a new and singular type of progression*, presented to the St. Petersburg Academy in 1771 [E476], Euler included several tables of data. For instance, with fifteen people, removing every fourth one gives the following order of removal:

$$4, 8, 12, 1, 6, 11, 2, 9, 15, 10, 5, 3, 7, 14, 13$$

He then analyzed the general problem to develop a recursive procedure for determining the number of the last person removed. In many cases the recursive step is just adding the number skipped to the previous answer.

To demonstrate that the procedure is feasible, Euler gave the computations to show that if there are 5000 people and every ninth person is removed, then the last one standing is number 4897.

Note: Euler publications are cited below by their Eneström number. All are reprinted in *Leonhard Euleri Oera Omnia*, abbreviated *OO*. Most are available electronically at *The Euler Archive*, http://eulerarchive.org, which also links to some English translations.

References

A. G. Andrews, chapter on partitions in *The History of Combinatorics*, R. J. Wilson ed., in preparation.

AE. G. Andrews and K. Eriksson, *Integer Partitions*, Cambridge University Press, 2004.

B. J. Bell, Euler and the pentagonal number theorem, Mathematics ArXiv HO/0510054, 2005.

D. L. Dickson, *History of the Theory of Numbers, vol. 2*, Carnegie Institution, Washington 1920. (Reprinted: Dover, 2005.)

E101. L. Euler, *Introductio in Analysin Infinitorum, v. 1*, 1748. Reprinted in *OO*, I.8. English translation: *Introduction to Analysis of the Infinite: Book I*, John Blanton trans., Springer-Verlag, 1998.

E158. L. Euler, Observationes analyticae variae de combinationibus, *Commentarii academiae scientiarum Petropolitanae* **13** (1751) 64–93. Reprinted in *OO*, I.2, 163–193.

E191. —, De partitione numerorum, *Novi Commentarii academiae scientiarum Petropolitanae* **3** (1753) 125–169. Reprinted in *OO*, I.2, 254–294.

E201. —, Calcul de la probabilité dans le jeu de rencontre, *Mémoires de l'académie des sciences de Berlin* **7** (1753) 255–270. Reprinted in *OO*, I.7, 11–25.

E309. —, Solution d'une question curieuse que ne pariot soumise à aucune analyse, *Mémoires de l'Académie Royale des Sciences et Belles Lettres, Année 1759, vol.15*, Berlin 1766, 310–337. Reprinted in *OO*, I.7, 26–56.

E394. —, De partitione numerorum in partes tam numero quam specie datas, *Novi Commentarii academiae scientiarum Petropolitanae* **14** (1770) 168–187. Reprinted in *OO*, I.3, 131–147.

E476. —, Observationes circa novum et singulare progressionum genus, *Novi Commentarii academiae scientiarum Petropolitanae* **20** (1776) 123–139. Reprinted in *OO*, I.7, 246–261.

E530. —, Recherches sur un nouvelle espèce de quarrés magiques, *Verhandelingen uitgegeven door het zeeuwsch Genootschap der Wetenschappen te Vlissingen* **9** (1782) 85–239. Reprinted in *OO*, I.7, 291–392.

E575. —, De mirabilibus proprietatibus unciarum, quae in evolutione binomii ad potestatem quamcunqua evecti occurrunt, *Acta Academiae Scientarum Imperialis Petropolitinae* **5** (1784) 74–111. Reprinted in *OO*, I.15, 528–568.

E584. —, De insignibus proprietatibus unciarum binomii ad uncias quorumvis polynomiorum extensis, *Acta Academiae Scientarum Imperialis Petropolitinae* **5** (1785) 76–89. Reprinted in *OO*, I.15, 604–620.

E709. —, De evolutione potestatis polynomialis cuiuscunque $(1 + x + x^2 + x^3 + x^4 + \text{ etc.})^n$, *Nova Acta Academiae Scientarum Imperialis Petropolitinae* **12** (1801) 47–57. Reprinted in *OO*, I.16, 28–40.

E738. —, Solutio quaestionis curiosae ex doctrina combinationum, *Mémoires de l'Académie des Sciences de St.-Pétersbourg* **3** (1811) 57–64. Reprinted in *OO*, I.7, 435-440.

E795. —, De quadratis magicis, *Commentationes arithmeticae* **2** (1849) 593–602. Reprinted in *OO*, I.7, 441–457.

EG. —, letter to Christian Goldbach, in *Correspondance Mathématique et Physique de Quelques Célèbres Géomètres du XVIIIème Siècle*, P. H. Fuss ed., 1845.

J. G. Jelliss Knight's Tour Notes, http://home.freeuk.net/ktn/

KS. D. Klyve and L. Stemkoski, Graeco-Latin Squares and a Mistaken Conjecture of Euler, *College Mathematics Journal* **37** (2006) 2–15.

dM. A. de Moivre, *The Doctrine of Chances: A Method of Calculating the Probabilities of Events in Play*, 1718. (Reprinted: American Mathematical Society, 2000.)

M. G. Mullen, A Candidate for the "Next Fermat Problem," *The Mathematical Intelligencer* **17** (1995) 18–22.

W. H. Wilf, *generatingfunctionology*, Academic Press, 1990.